*Malaysian Journal of Science* **35** (1): 37 – 43 (2016)

# AN APPROXIMATE SOLUTION OF TWO DIMENSIONAL NONLINEAR VOLTERRA INTEGRAL EQUATION USING NEWTON-KANTOROVICH METHOD

**Hameed Husam Hameed** [a,b*], **Z. K. Eshkuvatov** [c,d], **N.M.A. Nik Long** [a,c]

[a]Department of Mathematics, Faculty of Science, Universiti Putra Malaysia (UPM).
[b]Technical Institute of Alsuwerah, The Middle Technical University, Iraq.
[c] Institute for Mathematical Research, (UPM).
[d] Faculty of Science and Technology , Universiti Saina Islam Malaysia (USIM).
[*]hameedmath71@yahoo.co.in

**ABSTRACT** This paper studies the method for establishing an approximate solution of nonlinear two dimensional Volterra integral equations (NLTD-VIE). The Newton-Kantorovich (NK) suppositions are employed to modify NLTD-VIE to the sequence of linear two dimensional Volterra integral equation (LTD-VIE). The proper-ties of the two dimensional Gauss-Legenre (GL) quadrature fromula are used to abridge the sequence of LTD-VIE to the solution of the linear algebraic system. The existence and uniqueness of the approximate solution is demonstrated, and an illustrative example is provided to show the precision and authenticity of the method.

(**Keywords**: Newton-Kantorovich method, nonlinear operator, two dimensional Volterra integral equation, two dimensional Gauss-Legendre formula.)

## INTRODUCTION

Nonlinear two dimensional integral (NLTD) equations of the second kind have been exploited in several areas, including non homogeneous elasticity and electrostatics (Sankar T. S.& Fabrikant V. I.,1983). , contact problems for bodies with complex features (Aleksandrov V. M.& Manzhirov A. V. 19873)and (Manzhirov A. V.,1987), radio wave propagation (Soloviev O. V.,1998), as well as many physical, mechanical and biological phenomena. To date, many approximate methods have been operated and tested to achieve the solution of one-dimensional integral equations (Karoui A. & Jawahdou A., 2010; Maleknejad K. et al., 2011; Ezquerro J.A.at al., 2012; Bahyrycz A. et al., 2014; Mosleh M. & Otadi M.,2015; Chen Z.& Jiang W.,2015). However, confined research effort has been exerted to solve two-dimensional integral equations. A two-dimensional differential transform for double integrals has been promoted to solve NLTD-VIE (Tari A.et al., 2009). The piecewise constant two-dimensional block-pulse functions and their operational matrices have been invested for solving mixed NLTD Volterra-Fredholm integral equations of the first kind (Maleknejad K. & Mahdiani K.,2011). Two-dimensional orthogonal triangular functions have been exploited in (Maleknejad K.& JafariBehbahani Z.,2012) for solving non-linear mixed type Volterra-Fredholm integral equations. The approximate solution of a class of two dimensional nonlinear Volterra integral equations is given in (Nemati S. et al.,2013)by utilizing the properties of two-dimensional shifted Legendre functions to reduce the solution of the integral equation to the solution of a system of non-linear algebraic equations. In this study, we consider the NLTD-VIE of the second kind.

$$u(t,x)-\int_a^t\int_c^x K(t,x,y,z)G(y,z,u(y,z))dydz = f(t,x),\ (t,x)\in[a,b]\times[c,d], \quad (1)$$

where $u(t,x)\in\Omega_1$ is unknown function, $f(t,x)\in\Omega_1$ is presumed function, and $\Omega_1 = C_{[a,b]\times[c,d]}$, the kernel $K(t,x,y,z)$ is given smooth function and defined in $\Omega_1\times\Omega_2$, where $\Omega_2 = C_{[v_1,v_2]\times[v_3\times v_4]}$ and the nonlinear function $G(t,x,u(y,z))$ is continuous function which is defined in $\Omega_2\times(-\infty,\infty)$. The remainder of this paper is organized as follows. In Sections (II) we explain the use of the NK method to linearize the NLTD-VIE. In Section (III) the GL method is used to find the approximate solution of a sequence of the LTD-VIE. The theorem of existence and uniqueness of the solution is discussed in Section (IV). In Section (V) an example is provided to show the accuracy and efficiency of the method. Finally, Section (VI) concludes the key ideas of the proposed approximation method.

**LINEARIZE NLTD-VIE BY USING NK METHOD**

Let us use the operator form

$$P(u(t,x))=0 \tag{2}$$

to Eq. (1), we obtain

$$P(u(t,x))=u(t,x)-f(t,x) \\ -\int_a^t\int_c^x K(t,x,y,z)G(y,z,u(y,z))dydz=0, \tag{3}$$

then we use initial iteration of NK method of the form

$$P'(u_0(t,x))(u(t,x)-u_0(t,x))+P(u_0(t,x))=0, \tag{4}$$

to establish the approximate solution, where $u_0(t,x)$ is the initial guess and it may be any continuous function. The Frechet derivative of $P(u(t,x))$ at the initial guess $u_0(t,x)$ is appointed as

$$\begin{aligned} P'(u_0) &= \lim_{s\to 0}\frac{1}{s}\left[P(u_0+su)-P(u_0)\right] \\ &= \lim_{s\to 0}\frac{1}{s}\left[\frac{dP(u_0)}{du}su \right. \\ &\quad \left. +\frac{1}{2}\frac{d^2P}{du^2}\left(u_0+\theta su\right)s^2u^2\right] \\ &= \frac{dP(u_0)}{du}u,\ \theta\in(0,1). \end{aligned} \tag{5}$$

From Eqs (4) and (5) we obtain

$$\left.\frac{dP}{du}\right|_{u_0}\left(\Delta u(t)\right)=-P\left(u_0(t)\right), \tag{6}$$

where $\Delta u(t,x)=u_1(t,x)-u_0(t,x)$, and $u_0(t,x)$ is the initial function, then by establish the solution of Eq.(6) for $\Delta u(t,x)$ the derivative is computed as

$$\left.\frac{dP}{du}\right|_{u_0}=\lim_{s\to 0}\frac{1}{s}\left[P\left(u_0+su\right)-P(u_0)\right]$$

$$\begin{aligned} &= \lim_{s\to 0}\frac{1}{s}\left[su(t,x)-\int_a^t\int_c^x K(t,x,y,z)\right. \\ &\quad \left[G\left(y,z,u_0(y,z)+su(y,z)\right)\right. \\ &\quad \left.\left.-G\left(y,z,u_0(y,z)\right)\right]dydz\right], \end{aligned}$$

$$\left.\frac{dP}{du}\right|_{u_0}=u(t,x)-\int_a^t\int_c^x\left[K(t,x,y,z)\right. \\ \left.G'_u(y,z,u_0(y,z))u(y,z)dydz\right], \tag{7}$$

where $G'_u\left(y,z,u_0(y,z)\right)$ is the partial derivative of $G\left(y,z,u(y,z)\right)$ for $u(y,z)$. Therefore Eqs.(6) and (7) yield

$$\begin{aligned} &\Delta u(t,x)-\int_a^t\int_c^x\left[K(t,x,y,z)G'_u\left(y,z,u_0(y,z)\right)\right. \\ &\qquad \left.\Delta u(y,z)\right]dydz \\ &= f(t,x)+\int_a^t\int_c^x\left[K(t,x,y,z)\right. \\ &\qquad \left.G(y,z,u_0(y,z))\right]dydz-u_0(t,x), \end{aligned} \tag{8}$$

or

$$\Delta u(t,x)-\int_a^t\int_c^x K_0\left(t,x,y,z;u_0\right)\Delta u(y,z)dydz \\ =\mathbf{F}_0(t,x), \tag{9}$$

where

$$K_0(t,x,y,z;u_0)=\left[K(t,x,y,z)\right. \\ \left.G'(y,z,u_0(y,z))\right] \tag{10}$$

$$\mathbf{F}_0(t,x)=f(t,x)+\int_a^t\int_c^x\left[K(t,x,y,z)\right. \\ \left.G(y,z,u_0(y,z))\right]dydz-u_0(t,x). \tag{11}$$

We observe that Eq.(9) is a linear with respect to $\Delta u(t,x)$, and by solve it we find $u_1(t,x)=\Delta u(t,x)+u_0(t,x)$, then continuing this procedure, we get a sequence of approximate solution $u_m(t,x)$, $(m=2,3,\ldots)$ from the equation

$$P'\left(u_0(t,x)\right)\Delta u_m(t,x)+P\left(u_{m-1}(t,x)\right)=0 \tag{12}$$

that is same as the equation

$$\Delta u_m(t,x)-\int_a^t\int_c^x\left[K_0(t,x,y,z;u_0)\right. \\ \left.\Delta u_m(y,z)\right]dydz=\mathbf{F}_{m-1}(t,x), \tag{13}$$

where

$$\Delta u_m(t,x)=u_m(t,x)-u_{m-1}(t,x),\ \ m=2,3,\cdots, \tag{14}$$

and

$$\begin{aligned}\mathbf{F}_{m-1}(t,x) = f(t,x)+\int_a^t\int_c^x\big[K(t,x,y,z)\\ G(y,z,u_{m-1}(y,z))\big]dydz\\ -u_{m-1}(t,x).\end{aligned} \tag{15}$$

Solving Eq.(13) with respect to $\Delta u_m(t,x)$ we obtain a sequence of approximate solution $u_m(t,x)$.

## APPROXIMATE SOLUTION BY THE GL QUADRATURE METHOD

Introducing a grid

$$W=\left\{t_i,x_j : t_i=a+h_1\frac{b-a}{n_1}, x_j=c+h_2\frac{d-c}{n_2}\right\},$$

$i=1,2,\ldots,n_1$, $j=1,2,\ldots,n_2$, where $n_1$ and $n_2$ refer to the number of partitions in $[a,b]$ and $[c,d]$ respectively, Eq. (13) becomes

$$\begin{aligned}\Delta u_m(t_i,x_j)-\int_a^{t_i}\int_c^{x_j}\big[K_0(t_i,x_j,y,z;u_0)\\ \Delta u_m(y,z)\big]dydz\\ =\mathbf{F}_{m-1}(t_i,x_j),\end{aligned} \tag{16}$$

where

$$\begin{aligned}\mathbf{F}_{m-1}(t_i,x_j) = f(t_i,x_j)+\int_a^{t_i}\int_c^{x_j}\big[K(t_i,x_j,y,z)\\ G(y,z,u_{m-1}(y,z))\big]dydz\\ -u_{m-1}(t_i,x_j).\end{aligned} \tag{17}$$

The powerful technique to approximate the integration in Eq. (16) is GL quadrature formula. It is known thet Legendre polynomials $P_n(t)$ are orthogonal on $[-1,1]$ with weight $w=1$. Consider the GL quadrature formula for double integral

$$\begin{aligned}\int_{-1}^{1}\left(\int_{-1}^{1} f(x,u)du\right)dx=\sum_{i=1}^{n_1}\omega_{n_1 i}\left[\sum_{j=1}^{n_2}\omega_{n_2 j} f(s_{1i},s_{2j})\right.\\ \left. +R_{n_2}(s_{2j})\right]+\int_{-1}^{1}R_{n_1}(x)dx,\end{aligned} \tag{18}$$

where

$$\omega_{n_1 i}=\frac{2}{\left(1-s_{1i}^2\right)\left[P'_{n_1}(s_{1i})\right]^2},\ \sum_{i=1}^{n_1}\omega_{n_1 i}=2,$$

$$P_{n_1}(s_{1i})\equiv 0,\ i=1,2,\ldots,n_1, \tag{19}$$

$$\omega_{n_2 i}=\frac{2}{\left(1-s_{2i}^2\right)\left[P'_{n_1}(s_{2i})\right]^2},\ \sum_{i=1}^{n_2}\omega_{n_2 i}=2,$$

$$P_{n_2}(s_{2i})\equiv 0,\ i=1,2,\ldots,n_2, \tag{20}$$

are the corresponding weights or Christoffel numbers. $s_{1i}$ and $s_{2i}$ are roots of Legendre polynomials $P_{n_1}(t)$ and $P_{n_2}(t)$ over interval $[-1,1]$ respectively which have the error terms

$$R_{n_1}(f)=\frac{2^{2n_1+1}(n_1!)^4}{(2n_1+1)\left[(2n_1)!\right]^3} f^{2n_1}(\zeta),\ -1<\zeta<1.$$

$$R_{n_2}(f)=\frac{2^{2n_2+1}(n_2!)^4}{(2n_2+1)\left[(2n_2)!\right]^3} f^{2n_2}(\zeta),\ -1<\zeta<1.$$

The GL quadrature formula for arbitrary region $[a,b]\times[c,\ d]$ has form [15]

$$\begin{aligned}\int_a^b\int_c^d f(x,u)dudx\approx\left(\frac{b-a}{2}\right)\left(\frac{d-c}{2}\right)\\ \sum_{i=1}^{n_1}\left(\sum_{j=1}^{n_2}\omega_{n_{1i}}\omega_{n_{2j}} f(x_i,u_j)\right),\end{aligned} \tag{21}$$

where the knots $x_i=\left(\frac{b-a}{2}\right)s_{1i}+\left(\frac{b+a}{2}\right)$ and $u_i=\left(\frac{d-c}{2}\right)s_{2i}+\left(\frac{d+c}{2}\right)$. We propose a new idea that introduces a subgrids $\left(W_{n_1}\right)$ and $\left(W_{n_2}\right)$ of $l_1$ and $l_2$ Legendre knot points at each subintervals $[a,t_i]$ and $[c,x_j]$ respectively. that are included in the intervals $[a,b]$ and $[c,d]$ which appear in Eq.(28) that

$$\begin{aligned}\tau_{n_1 i}^{k_1}=\frac{t_i-a}{2}s_{k_1}+\frac{t_i+a}{2},\\ i=1,2,\ldots,n_1, k_1=1,2,\ldots,l_1\end{aligned} \tag{22}$$

$$\begin{aligned}\tau_{n_2 i}^{k_2}=\frac{x_j-a}{2}s_{k_2}+\frac{x_j+a}{2},\\ j=1,2,\ldots,n_2, k_2=1,2,\ldots,l_2\end{aligned} \tag{23}$$

where $\tau_{n_1 i}^{k_1} \neq t_i$ and $\tau_{n_2 i}^{k_2} \neq x_i$. Extending GL quadrature formula to the integral in both two subintervals $\lceil a, t_i \rceil$ and $\lceil c, x_j \rceil$ in Eq. (16), we get

$$\begin{aligned}
&\Delta u_m\left(\tau_{n_1 i}^{r_1}, \tau_{n_2 i}^{r_2}\right) - \left(\frac{t_i - a}{2}\right)\left(\frac{x_j - c}{2}\right) \\
&\quad \sum_{k_1=1}^{i}\sum_{k_2=1}^{j}\Big[K_0\left(\tau_{n_1 i}^{r_1}, \tau_{n_2 j}^{r_2}, \tau_{n_1 i}^{k_1}, \tau_{n_2 j}^{k_2}; u_0\right) \\
&\quad \Delta u_m\left(\tau_{n_1 i}^{k_1}, \tau_{n_2 j}^{k_2}\right)\omega_{n_1 k_1}\omega_{n_2 k_2}\Big] \qquad (24) \\
&\quad = \mathbf{F}_{m-1}\left(\tau_{n_1 i}^{r_1}, \tau_{n_2 j}^{r_2}\right),
\end{aligned}$$

$i = 1, 2, \ldots, n_1,\ j = 1, 2, \ldots, n_2$

$r_1 = 1, 2, \ldots, l_1, r_2 = 1, 2, \ldots, l_2$

where

$$\begin{aligned}
\mathbf{F}_{m-1}\left(\tau_{n_1 i}^{r_1}, \tau_{n_2 j}^{r_2}\right) &= f\left(\tau_{n_1 i}^{r_1}, \tau_{n_2 j}^{r_2}\right) \\
&+ \left(\frac{t_i - a}{2}\right)\left(\frac{x_j - c}{2}\right) \\
&\sum_{k_1=1}^{i}\Big[\sum_{k_2}^{j} K\left(\tau_{n_1 i}^{r_1}, \tau_{n_2 j}^{r_2}, \tau_{n_1 i}^{k_1}, \tau_{n_2 j}^{k_2}\right) \qquad (25) \\
&G\left(\tau_{n_1 i}^{r_1}, \tau_{n_2 j}^{r_2}, u_{m-1}\left(\tau_{n_1 i}^{k_1}, \tau_{n_2 j}^{k_2}\right)\right)\omega n_1 k_1 \omega n_2 k_2\Big] \\
&- u_{m-1}\left(\tau_{n_1 i}^{r_1}, \tau_{n_2 j}^{r_2}\right).
\end{aligned}$$

Eq.(25) is a linear algebraic system of $(n_1 \times n_2) \times (l_1 \times l_2)$ equations and $(n_1 \times n_2) \times (l_1 \times l_2)$ unknowns. If the non singularity is achieved of this system, then it has unique solution in terms of $\Delta u_m(t, x)$, $(m = 2, 3, \ldots)$. From eq.(14) it follows that

$$u_m(t, x) = \Delta_m(t, x) + u_{m-1}(t, x),\ m = 2, 3, \ldots \qquad (26)$$

## CONVERGENCE ANALYSIS

Using the general theorem of NK method and their applications to functional equations, we state the following theorem for successive approximations which are characterized by Eq. (13).

First, since $f(t, x)$, $u_0(t, x)$, $K(t, x, y, z)$, $G(\zeta)$, $G'(\zeta)$ and $G''(\zeta)$ are continuous in their domain of definitions, then they are bounded ([16], pp 33), such that

$$\begin{aligned}
&|f(t, x)| \le M_1, |u_0(t, x)| \le M_2, |K(t, x, y, z)| \le M_3, \\
&|G(t, x, u_0(t, x))| \le M_4, |G'(t, x, u_0(t, x))| \le M_5, \\
&|G''(t, x, u_0(t, x))| \le M_6.
\end{aligned}$$

Then, we use the majorant function [7]

$$\varphi(t) = \eta(t - t_0)^2 - (1 + \eta\xi)(t - t_0) + \xi, \qquad (27)$$

where $\eta$ and $\xi$ are nonnegative real number. Let $\eta_1 = M_3 M_6 (b - a)(c - d)$

**Theorem:** Let the operator $P(x) = 0$ in Eq. (3) is defined in $\Omega = \left\{u \in C_{[a,b]\times[c,d]} : \|u - u_0\| \le R\right\}$ and has a continuous second derivative in $\Omega_0 = \left\{u \in C_{[a,b]\times[c,d]} : \|u - u_0\| \le r\right\}$. If

1) The linear VIE in Eq. (13) has a resolvent kernel $\Gamma(t, x; \lambda)$ where $\|\Gamma\| \le M_3 M_5 e^{M_3 M_5 (b-a)(c-d)}$,

2) $|\Delta u| \le \dfrac{\xi}{1 + \eta\xi}$,

3) $|P''(x)| \le \eta_1$.

Then eq. (1) has a unique solution $u^*(t, x)$ in the closed ball $\Omega_0$ and the sequence $u_m(t, x),\ m \ge 0$ of successive approximation

$$\begin{aligned}
&\Delta u_m(t_i, x_j) - \int_a^{t_i}\int_c^{x_j} K_0(t_i, x_j, y, z; u_0)\Delta u_m(y, z)\,dydz \qquad (28) \\
&\quad = \mathbf{F}_{m-1}(t_i, x_j),
\end{aligned}$$

where $\Delta u_m(t, x) = u_m(t, x) - u_{m-1}(t, x)$ converges to the solution $u^*(t, x)$. The rate of convergence is given by

$$\|u^* - u_m\| \le \left(\frac{2}{1 + \eta\xi}\right)^m \left(\frac{1}{\eta}\right), m = 1, 2, \ldots \qquad (29)$$

**Proof:** It is shown that Eq.(3) is reduced to Eq. (9). Since Eq. (9) is a linear integral equation of second kind for $\Delta u(t, x)$, then it has a unique solution in term of $\Delta u(t, x)$ provided that its kernel $K_0(t, x, y, z; u_0)$ is continuous function. Hence the existence of $\Gamma_0$ is achieved.

To prove $\Gamma_0$ is bounded we need to find the resolvent kernel $\Gamma_0(t, x, y, z; u_0)$ of Eq. (9). Assume the integral operator $\mathbf{V}$ from $C_{[a,b]\times[c,d]} \to C_{[a,b]\times[c,d]}$ is given by

$$\mathbf{Z} = \mathbf{V}(\Delta u),$$
$$\mathbf{Z}(t,x) = \int_a^t \int_c^x \big[K_0(t,x,y,z;u_0) \Delta u(y,z)\big] dydz, \tag{30}$$

where $K_0(t,x,y,z;u_0)$ is defined in Eq.(10). According to Eq. (9), Eq. (30) can be written as

$$\Delta u - \mathbf{V}(\Delta u) = \mathbf{F}_0(t). \tag{31}$$

The solution $\Delta u^*$ of Eq. (31) is written in terms of $\mathbf{F}_0$ as

$$\Delta u^* = \mathbf{F} + \mathbf{B}(\mathbf{F}_0), \tag{32}$$

where $\mathbf{B}$ is an integral operator and can be represented as a power of $\mathbf{V}$ ([18], Theorem 1, pp. 378)

$$\mathbf{B}(\mathbf{F}_0) = I + \mathbf{V}(\mathbf{F}_0) + \mathbf{V}^2(\mathbf{F}_0) + \cdots + \mathbf{V}^n(\mathbf{F}_0) + \cdots, \tag{33}$$

and it is well known that the powers of $\mathbf{V}$ are also integral operators . In fact

$$\mathbf{Z}_n = \mathbf{V}^n,$$
$$\mathbf{Z}_n(t,x) = \int_a^t \int_c^x \big[K^n{}_0(t,x,y,z;u_0) \Delta u(y,z)\big] dydz, (n=1,2,\ldots), \tag{34}$$

where $K_o^n$ is the iterated kernel, Substituting Eq. (34) into Eq. (32) we obtain the solution of eq. (31) which is of the form

$$\Delta u^*(t,x) = \mathbf{F}(t,x) + \int_a^t \int_c^x [\Gamma_0(t,x,y,z;u_0) \mathbf{F}_0(y,z)\big] dydz, \tag{35}$$

where

$$\Gamma_0(t,x,y,z,u_0) = \sum_{j=0}^{\infty} K_0^{j+1}(t,x,y,z;u_0), \tag{36}$$

where $\Gamma_0(t,x,y,z,u_0)$ is the resolvent kernel. Next, we state that the series in Eq.(35) is convergent uniformly for all $t \in [a,b]$ and $x \in [c,d]$. Since

$$\begin{aligned} |K_0(t,x,y,z;u_0)| &= |K(t,x,y,z)||G'(y,z,u_0(y,z))| \\ &\le M_3 M_5. \end{aligned} \tag{37}$$

Let $M = M_3 M_5$, then by mathematical induction we obtain

$$\begin{aligned} |K_0^2(t,x,y,z,u_0)| &\le \int_a^t \int_c^x |K_0(t,x,y,z;u_0) K_0(t,x,y,z;u_0)| dydz \\ &\le \frac{M^2(b-a)(d-c)}{(1)!}, \end{aligned}$$

$$\begin{aligned} |K_0^3(t,x,y,z,u_0)| &\le \int_a^t \int_c^x |K_0(t,x,y,z;u_0) K_0^2(t,x,y,z;u_0)| dydz \\ &\le \frac{M^3(b-a)^2(d-c)^2}{(2)!}, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} |K_0^n(t,x,y,z,u_0)| &\le \int_a^t \int_c^x |K_0(t,x,y,z;u_0) K_0^{n-1}(t,x,y,z;u_0)| dydz \\ &\le \frac{M^2(b-a)^{n-1}(d-c)^{n-1}}{(n-1)!}, \end{aligned}$$

then

$$\begin{aligned} \|\Gamma_0\| = \|\mathbf{B}(\mathbf{F}_0)\| &\le \sum_{j=0}^{\infty} |K_0^{j+1}(t,x,y,z;u_0)|, \\ &\le \sum_{j=0}^{\infty} M^{j+1} \frac{(b-a)^j (c-d)^j}{j!}, \\ &= M \sum_{j=0}^{\infty} M^j \frac{(b-a)^j (c-d)^j}{j!}, \\ &= M e^{M(b-a)(d-c)}. \end{aligned}$$

Therefore, the infinite series in Eq. (36) for $\Gamma_0(t,x,y,z;u_0)$ converges uniformly for all $t \in [a,b]$ and $x \in [c,d]$. Now, we prove $\|P''(u)\| \le \eta_1$ for all $u(t,x) \in \Omega_0$. It is shown that the second derivative $P''(u_0)(u)$ of nonlinear operator $P(u)$ at the point $u_0$ refres to the bilinear operator i.e. $P''(u_0)(u) = B(u,u_0)$ ([18], pp. 506). By the definition of the second derivative, $P''(u_0)(u)$ has the form

$$P''(u_0)u = \lim_{s\to 0}\frac{1}{s}\left[P'(u_0+su)-P'(u_0)\right],$$
$$= \lim_{s\to 0}\frac{1}{s}\left(\frac{d^2P}{du^2}(u_0)s\bar{u}+\frac{1}{2}\frac{d^3P}{du^3}(u_0+\theta s\bar{u})s^2\bar{u}^2\right),$$
$$= \left.\frac{d^2P}{du^2}\right|_{u_0}\bar{u},$$

then the norm of $\left\|\frac{d^2P}{du^2}\right\|$ has the estimate

$$\left\|\frac{d^2P}{du^2}\right\| = \max_{\|u\|\le 1,\|\bar{u}\|\le 1}\left|\int_a^t\int_c^x\left[K(t,x,y,z)\, G''(y,z,u_0(y,z))\, u(y,z)\bar{u}(y,z)\right]dydz\right|$$
$$\le M_3M_6(b-a)(c-d).$$

Therefore, the second derivative exist is bounded, that implies $u^*(t,x)$ is the unique solution of operator equation (3) ([18], Theorem 6, pp. 532).
The rate of convergence is given by [17]

$$\|u^*-u_m\| \le \left(\frac{2}{1+\eta\xi}\right)^m\left(\frac{1}{\eta}\right),\ m=1,2,\ldots \qquad (38)$$

## NUMERICAL RESULTS

Our aim in this section to show the ability of the NK method for solving the nonlinear integral equations of Volterra type by giving an example. For computing the result in each table. We use MATLAB VRa 2008.

**Example**: consider the following integral equation

$$u(t,x)-\int_0^t\int_0^x(y^2+e^{-2z})u^2(y,z)dydz$$
$$= x^2e^t+\frac{1}{14}x^7-\frac{1}{14}x^7e^{2t}-\frac{1}{5}x^5t, \qquad (39)$$
$$t\in[0,1]\times[0,1].$$

**Table 1**. Numerical result for Eq. (39).

$$n_1=n_2=2, l_1=l_2=5,$$
$$h_1=h_2=0.5, u_0(t,x)=xt^2.$$

| m. | $\varepsilon_u$ |
|---|---|
| 1. | 0:032626108681354 |
| 2. | 0:010379355154074 |
| 3. | 0:003531303306024 |
| 4. | 0:001225693869725 |
| 5. | 4:282749605E-004 |
| 10. | 2:266046910E -006 |
| 20 | 6:3682836782E -011 |

**Table 2**. Numerical result for Eq. (39).

$$n_1=n_2=2, l_1=l_2=5,$$
$$h_1=h_2=0.5, u_0(t,x)=\sqrt{xt}.$$

| m. | $\varepsilon_u$ |
|---|---|
| 1. | 0:032626108681354 |
| 2. | 0:051110800360088 |
| 3. | 0:025183884200234 |
| 4. | 0:012873335033743 |
| 5. | 0:006694764010190 |
| 10. | 2:732561754E - 004 |
| 20 | 4:722354158E - 007 |
| 34 | 6:416467357E -011 |

Table 1 shows that few iterations are needed for $u_m(t)$ to be very close to $u^*(t)$, while Table 2 refers that if we choose another initial guess that far from the exact solution, we need more iteration to the good approximate solution. Notations used here are: $n_1$ and $n_1$ are the number of partitions on $[a,b]$ and $[c,d]$ respectively, $l_1$ and $l_2$ are the number of subpartitions on $(a,t_i)$ and $(c,x_j)$ respectively, $i=1,2,\ldots,n_1$, $j=1,2,\ldots,n_2$ where $m$ is the number of iterations and

$$\varepsilon_u = \max_{t\in(0,1]}\left|u_m(t)-u^*(t)\right|. \qquad (40)$$

## CONCLUSION

In this paper, the NK method is offered to solve the NLTD-VIE. We suggested a new idea by introducing a subgrid of collocation points $\tau_{n_1i}^{k_1}$ and $\tau_{n_2j}^{k_2}$, $i=1,2,\ldots,n_1$, $j=1,2,\ldots,n_2$ and $k_1=1,2,\ldots,l_1$, $k_2=1,2,\ldots,l_2$ which are contained in $[a,t_i]$ and $[c,x_j]$. The theorem of existence and uniqueness of approximate solution is introduced based on the general theorems of Kantorovich. The numerical example is given to show the efficiency of the method.

## ACKNOWLEGMENT

This work was supported by Universiti Putra Malaysia under Research Grand Putra, 2014. Project code is GP-I/2014/9442300. Authors are grateful for sponsor and financial support from the Research Management Center, Universiti Putra Malaysia.

## REFERENCES

Aleksandrov V. M., Manzhirov A. V. (1987). Two-dimensional integral equations in applied mechanics of deformable solids. J. Appl. Mech. Tech. Phys. 5 146-152.

Bahyrycz A, Brzdek J., Lesniak Z. (2014). On approximate solutions of the generalized Volterra integral equation. Nonlinear Analysis: Real World Applications. 20 5966.

Chen Z., Jiang W. (2015). An efficient algorithm for solving nonlinear Volterra-Fredholm integral equations. Applied Mathematics and Computation. 259 614-619.

Elliott D., Johnston P. R., Johnston B. M. (2011). Estimates of the Error in Gauss-Legendre Quadrature for Double Integrals. Journal of Computational and Applied Mathematics. 236, 1552-1561.

Ezquerro J.A., Gonzalez D., Hernandez M.A. (2012). A variant of the Newton-Kantorovich theorem for nonlinear integral equations of mixed Hammerstein type. Applied Mathematics and Computation. 218 9536-9546.

Hameed H. H., Eshkuvatov Z. K., Ahmedov A., NikLong N.M.A. (2015). On Newton-Kantorovich Method for Solving the Nonlinear Operator Equa-tion. Abstract and Applied Analysis. Volume Article ID 219616. 12 pages.

Kantorovich L.V., Akilov G.P. (1982). Functional Analysis. Pergamon press Ltd.and Nauka publisher.

Karoui A., Jawahdou A. (2010). Existence and approximate L p and contin-uous solutions of nonlinear integral equations of the Hammerstein and Volterra types. Applied Mathematics and Computation. 216 2077-2091.

Maleknejad K., Jafari Behbahani Z. (2012) Applications of two-dimensional triangular functions for solving nonlinear class of mixed Volterra-Fredholm integral equations. Mathematical and Computer Modelling 55 18331844

Maleknejad K., Mahdiani K. (2011). Solving nonlinear mixed Volterra-Fredholm integral equations with two dimensional block-pulse functions using direct method. Commun Nonlinear Sci Numer Simulat. 16 3512-3519.

Maleknejad K., Torabi P., Mollapourasl R. (2011). Fixed point method for solving nonlinear quadratic Volterra integral equations, Computers and Mathematics with Applications. 62 2555-2566.

Manzhirov A. V. (1987). Contact problems of the interaction between viscoelas-tic foundations subject to ageing and systems of stamps not applied simultaneously. Prikl. Matem. Mekhan.. 4 523-535.

Mosleh M., Otadi M. (2015). Least squares approximation method for the solution of Hammerstein-Volterra delay integral equations. Applied Mathematics and Computation. 258 105-110.

Nemati S., Lima P.M., Ordokhani Y. (2013). Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials. Journal of Computational and Applied Mathematics. 242 53-69.

Sankar T. S., Fabrikant V. I. (1983). Investigations of a two-dimensional integral equation in the theory of elasticity and electrostatics. J. Mec. Theor.Appl. 2 285-299.

Soloviev O. V. (1998). Low-frequency radio wave propagation in the earth-ionosphere waveguide disturbed by a large-scale three-dimensional ir-regularity. Radiophysics and Quantum Electronics. 41 392-402.

Tari A., Rahimi M.Y., Shahmorad S., Talati F. (2009). Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method, Journal of Computational and Applied Mathematics. 228 70-76.

Zeidler E., (1995). Applied Functional Analysis Main Principles and Their Applications. Springer-Verlag New York, Inc.